a review of Rigidity of a family of spherical conical metrics by Zhu, Xuwen

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Zhu, Xuwen

Rigidity of a family of spherical conical metrics. (English) Zbl 07179060

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The study of constant curvature metrics with singularities has a long and affluent history, a lot of interesting questions being still not completely settled. The following singular uniformization problem is not settled either.

Problem 1. Given a compact Riemann surface M, a collection of distinct points

$$\mathfrak{p} = \{p_1, \dots, p_k\} \subset M$$

and a collection of positive real numbers

$$\beta_1, \dots, \beta_k$$

is it possible to find a metric g on M with constant curvature and with conic singularities abiding by prescribed cone angles $2\pi\beta_j$ at the points p_j?

The sign of its curvarture is determined by the so-called *conic* Gauss-Bonnet formula

$$\frac{1}{2\pi}\int_M K dA = \chi\left(M, \overrightarrow{\beta}\right) := \chi\left(M + \sum_{j=1}^{k}(\beta_j - 1)\right)$$

In case of $\chi\left(M, \overrightarrow{\beta}\right) \leq 0$, the uniqueness and existence of such solutions were established in [Zbl 0657.30033]. In the spherical case of $K = 1$ with all the cone angles less than 2π, the existence problem was settled in [Zbl 0724.53023], while the uniqueness problem was decided in [Zbl 0806.53012]. Deformation theory for the above cases without any restriction on the position of cone points has been investigated in [Zbl 06741332], where it was shown that the metrics are of smooth dependence on cone angles and positions.

In case of $K = 1$ with at least one of the cone angles bigger than 2π, the story is much more challenging. Recently it was established in [Zbl 06992339] that when $M = \mathbb{S}^2$ the cone angles are constrained by the following linear inequalities

$$d_1\left(\overrightarrow{\beta} - \overrightarrow{1}, \mathbb{Z}^k_{\text{odd}}\right) \geq 1$$

and the existence was settled there in case that the strict inequality holds, while the boundary cases have been considered in [Zbl 1404.53047, `arXiv:1706.04608`, `https://www.math.ucdavis.edu/~kapovich/EPR/covers.pdf`]. Mondello and Panov [Zbl 07091755] have shown that when $M \neq \mathbb{S}^2$, the condition $\chi\left(M, \overrightarrow{\beta}\right) > 0$ is sufficient. In either case, one is unable to specify the marked conformal class, in other words, the location of the points $\mathfrak{p}$.

This paper considers the deformation of the following metrics with four conical points on $\mathbb{S}^2$

$$2\pi\beta = (\alpha, \beta, \alpha+\beta, 4\pi)\,,\ \alpha, \beta, \alpha+\beta \notin \{2\pi\mathbb{Z}, \pi + 2\pi\mathbb{Z}\}$$

and demonstrates that there is local rigidity in the location of cone points, where *local rigidity* means that, for any small deformation of this metric inside the class of spherical conical metrics with the same cone angle combination, the underlying pointed conformal structure does not change. The author's argument is highly synthetic, using elementary spherical geometry. The above angle combination was discussed in [Zbl 1308.30050, Examaple 4.7], in which it was shown that if such a metric has reducible monodromy, then the positions of cone points with the first three angles determine the position of the 4th point and there is a real 1-parameter family of such metrics where the parameter comes from the scaling of the character 1-form. The above angle combination was investigated in detail as the solutions to Fuchsian equations [Zbl 1397.34153; JFM 28.0360.03].

The geometric rigidity discussed in the paper has appeared in the case of a spherical football with noninteger cone angles. A spherical metric with cone angles $\alpha, \beta, \alpha+\beta, 4\pi$ can be obtained by gluing together two spherical footballs along a slit of length t. The author introduces the notion of a *triangulated metric*. The space of triangulated metrics is parametrized by six independent lengths as long as they abide by spherical triangle inequalities. If such metrics are restricted to four fixed cone angles, then the space of such metrics is two-dimensional for a generic angle set $\overrightarrow{\beta}$, meaning that the neighborhood of triangulated metrics gives all nearby spherical conical metrics with the same cone angle data. It is shown that the glued footballs are the only possible triangulated metrics under perturbation, meaning that all such metrics form a one-dimensional space, which is no other than the local rigidity in the geometric sense.

The classical argument in [Zbl 0697.53037] showed that the only possible configuration in this case is when all the geodesics connecting two cone points are of length π. The perturbation argument of [Zbl 06741332] established that the spherical football is the only metric with rigidity in case that all cone angles are less than 2π. However, when there are more than three cone points with some angles bigger than 2π, rigidity is not easy to obtain, and the main result of this paper, giving a family of explicit metrics, is of much interest. If the metrics on a football with angle $2\pi\alpha$ are written in geodestic coordinates as

$$dr^2 + \alpha^2 \sin^2 r d\theta^2$$

then $\cos r$ is an eigenfunction of its Laplacian with eigenvalue 2. Furthermore, when two footballs not necessarily with the same angle glue together, this eigenfunction is also to glue so as to render a global one, such partial rigidity in cone positions being expected as a result of the obstruction in solving the curvature equation [`arXiv:1906.09720`].

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`53Cxx` Global differential geometry

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